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Trace formulas and the Conley–Zehnder index

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The topic of this paper is a generalization of the Conley–Zehnder index for periodic trajectories of a classical Hamiltonian system (Q, ω, H) from cotangent bundles $Q = T^*\mathbb{R}^n$ to arbitrary symplectic manifolds. It is precisely this index that occurs as a "Maslov phase" in the trace formulas by Gutzwiller and Duistermaat–Guillemin. In the course of constructing the index, a survey and several new formulas for Maslov's theory of the Lagrangian Grassmannian are presented.

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Introduction

Consider a classical mechanical system (Q, ω, H), where Q is a symplectic manifold, ω its symplectic two-form and $H \in C^\infty(Q, \mathbb{R})$ a Hamiltonian. Assume that $\gamma : \mathbb{R} \to Q$ is a periodic solution curve of period T and energy E for the Hamiltonian vector field X_H on Q. The closed orbit is called *regular* if the (linear) Poincaré map $P(T)$ has no unit eigenvalue. From the implicit function theorem it follows that regular periodic orbits always come in one-parameter families. Their "orbit cylinder" is a symplectic submanifold of Q which is transversal to all energy surfaces $H^{-1}(E)$. We refer to ref. [1] for a detailed proof, which also shows that orbit cylinders are stable under small perturbations of the Hamiltonian.

Conley and Zehnder [3] have defined an index $\mathrm{ind}_{CZ}(\gamma)$ for regular periodic orbits in $T^*\mathbb{R}^n$, generalizing the usual Morse index for closed geodesics on a Riemannian manifold. Roughly speaking, the index measures how often neighbouring trajectories of the same energy wind round the orbit. It is stable under deformations of the orbit as long as the regularity assumption is not violated. In particular, all members of the orbit cylinder have the same index.

As we will see below, the Conley–Zehnder index admits a natural extension to arbitrary symplectic manifolds. The construction will only depend on the choice of a homotopy class of Lagrangian subbundles L of TQ along the orbit. Such

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a choice is often dictated by the particular system under study, and is natural, e.g., for cotangent bundles $Q = T^*X$ or if the orbit is contractible. The index $\mathrm{ind}(\gamma, L)$ is characterized by the following two properties:

– The index is stable under small perturbations of the system (as long as the orbit remains regular).

– Whenever there is an invariant Lagrangian subbundle M of TQ along the orbit, $\mathrm{ind}(\gamma, L)$ is the (Maslov) intersection number of M with L.

Regular periodic trajectories of the bicharacteristic flow play an essential role in the so-called trace formulas by Duistermaat–Guillemin [8] and Gutzwiller [10] and various generalizations [2,12]. Let us briefly recall the content of these formulas. The Duistermaat–Guillemin theorem deals with positive elliptic first order pseudo-differential operators $\hat{H}$ on compact manifolds X. It states that the trace of the corresponding unitary group (which is the Fourier transform of the spectral density) has singularities at the periods T of periodic trajectories for the principal symbol $H \in C^\infty(T^*X, \mathbb{R})$. Moreover, under certain "cleanness" assumptions on the flow, the residue at a nonzero period corresponding to the orbit γ is given by the expression

$$\lim_{t \to T} (t - T) \operatorname{Tr}(\mathrm{e}^{-\mathrm{i}\hat{H}t}) = \frac{T_\gamma^\sharp}{2\pi} \frac{\mathrm{e}^{-\mathrm{i}\alpha T}}{\sqrt{|\det(I - P_\gamma)|}} \exp(-\tfrac{1}{2}\mathrm{i}\pi\sigma_\gamma), \tag{1}$$

where $T_\gamma^\sharp$ is the fundamental period of the orbit, P_γ its Poincaré map, α the mean value of the subprincipal symbol along γ and $\sigma_\gamma \in Z$ a suitable Maslov phase.

The (nonrigorous) semiclassical trace formula due to Gutzwiller stands at the outset for the theoretical study of "quantum chaos". According to this formula, the spectral density $g(E) = \sum \delta(E - E_i(\hbar))$ for Schrödinger operators with discrete spectrum and sufficiently chaotic classical counterpart is approximately (for $\hbar \to 0$) the sum of a non-oscillating "Thomas–Fermi contribution" and an oscillating part (yielding energy levels). The former corresponds to the rule that each quantum state occupies a volume $(2\pi\hbar)^n$ in classical phase space. The latter is a sum over the periodic orbits of the system, the contribution of γ being

$$g_\gamma(E) = \frac{T_\gamma^\sharp}{2\pi} \frac{1}{\sqrt{|\det(I - P_\gamma)|}} \mathrm{e}^{(\mathrm{i}/\hbar)S(E)} \exp(-\tfrac{1}{2}\mathrm{i}\pi\sigma_\gamma), \tag{2}$$

where $S(E) = \oint \theta_{T^*X}$ denotes the action integral. Rigorous versions of the Gutzwiller formula can be derived with the help of suitable smoothing and localizing operators, see refs. [2,16,19].

One of the goals of this work is to elucidate the geometric meaning of the Maslov phase in these formulas. We shall prove that in both cases, σ_γ is nothing but the Conley–Zehnder index $\mathrm{ind}(\gamma, V)$, V being the vertical polarization. For (1) this is equivalent to a result by Duistermaat [7]. It also incorporates a recent observation by Robbins [18], who has shown that for hyperbolic periodic orbits, σ_γ simply counts how often the stable and unstable manifolds wind round γ.

The paper is divided into four parts. The first section gives a survey of various indices associated with the symplectic group $\mathrm{Sp}(E)$ and the Lagrangian Grassmannian $\Lambda(E)$. We shall start from Dazord's [5] construction of the intersection number [7] for a pair of curves of Lagrangian subspaces, built on Kashiwara's definition of the signature of a Lagrangian triplet.

This approach does not require transversality at the endpoints, and may be used to simplify the construction [9,15] of Leray's index in the nontransversal case. The second section is concerned with the index of a periodic orbit. The third section contains the proof of $\sigma_\gamma = \mathrm{ind}(\gamma, V)$. Finally, the appendix gives a formula for the signature which leads to a very simple proof of the crucial cocycle identity.

It should be emphasized that this contribution is closely linked with the paper [7] by Duistermaat, which contains further information on the intersection theory of Lagrangian curves and an extensive discussion of its relation to Morse theory.

1. The intersection number of Lagrangian subspaces

Let (E, ω) be a real symplectic vector space of dimension $2n$ and let $\Lambda(E)$ be its Lagrangian Grassmannian, i.e. the set of Lagrange subspaces of E. Consider the action of the symplectic group $\mathrm{Sp}(E)$ on the set $\Lambda(E)^3$ of ordered Lagrangian triplets (L_1, L_2, L_3). It is clear that the dimensions of the intersections are invariant under this action. Another independent invariant is the so-called *signature of a Lagrangian triplet* discovered by Hörmander [13] and Kashiwara [15]. Together, these invariants completely specify the relative position of three Lagrangian subspaces up to symplectic transformations. For $(L_1, L_2, L_3) \in \Lambda(E)^3$, the signature $s(L_1, L_2, L_3) \in \mathbb{Z}$ is defined by

$$s(L_1, L_2, L_3) = \mathrm{sgn}\,(Q(L_1, L_2, L_3)), \tag{3}$$

where $Q(L_1, L_2, L_3)$ is the quadratic form

$$\begin{aligned} &Q(L_1, L_2, L_3) : L_1 \oplus L_2 \oplus L_3 \to \mathbb{R}, \\ &(x_1, x_2, x_3) \mapsto \omega(x_1, x_2) + \omega(x_2, x_3) + \omega(x_3, x_1). \end{aligned} \tag{4}$$

It is immediate from the definition that the signature $s : \Lambda(E)^3 \to \mathbb{Z}$ is invariant under symplectic transformations and antisymmetric under exchange of two of the L's. Let us list some less trivial properties:

Proposition 1.

(1) Cocycle identity:

$$s(L_2, L_3, L_4) - s(L_1, L_3, L_4) + s(L_1, L_2, L_4) - s(L_1, L_2, L_3) = 0.$$

(2) Reduction lemma: For arbitrary subspaces K of $L_1 \cap L_2 + L_2 \cap L_3 + L_3 \cap L_1$,

$$s(L_1, L_2, L_3) = s(L_1^K, L_2^K, L_3^K),$$

where L_i^K denotes the image of L_i under the symplectic reduction $(K + K^\omega) \to E^K := (K + K^\omega)/(K \cap K^\omega)$.

(3) The signature runs through all integers between $-\frac{1}{2}\dim E^F$ and $+\frac{1}{2}\dim E^F$, where $F = (L_1 \cap L_2) + (L_2 \cap L_3) + (L_3 \cap L_1)$. Consequently, $s(L_1, L_2, L_3) + \dim(L_1 \cap L_2) + \dim(L_2 \cap L_3) + \dim(L_3 \cap L_1) + n$ is an even number.

(4) The orbits of the action of $\mathrm{Sp}(E)$ on $\Lambda(E)^3$ are completely determined by $\dim(L_1 \cap L_2 \cap L_3)$, $\dim(L_1 \cap L_2)$, $\dim(L_2 \cap L_3)$, $\dim(L_3 \cap L_1)$ and $s(L_1, L_2, L_3)$: If these five numbers coincide for two triplets, they lie on the same orbit.

(5) The signature is locally constant on the set of all triplets with given dimensions of intersections.

The proofs of the first two properties can be found, e.g., in ref. [15]. For the case $L_i \cap L_j = \{0\}$, properties 3 and 4 are proven in ref. [11] and property 5 becomes obvious. The general cases follow by repeated use of the reduction lemma.

Consider now two continuous paths $L_1, L_2 : [a, b] \to \Lambda(E)$, where $a \leq b$. Assume first that there is some $M \in \Lambda(E)$ which is transversal to all $L_i(t)$. We claim that the expression

$$[L_1 : L_2] := \tfrac{1}{2}\big[s(L_1(a), L_2(a), M) - s(L_1(b), L_2(b), M)\big] \tag{5}$$

is independent of the choice of M. Indeed, the cocycle identity gives

$$\begin{aligned} &s(L_1(t), L_2(t), M) - s(L_1(t), L_2(t), M') \\ &= s(L_2(t), M, M') - s(L_1(t), M, M'), \end{aligned}$$

which according to proposition 1.5 is independent of t if M, M' are transversal to all $L(t)$. Thus, replacing M by M' changes both terms in (5) by the same amount.

In the general case, we choose a sufficiently fine partition $a = t_0 \leq \cdots \leq t_k = b$ and Lagrangian subspaces M_ν such that M_ν is transversal to all $L_i(t)$ with $t_{\nu-1} \leq t \leq t_\nu$, $i = 1, 2$, and define the *intersection number* $[L_1 : L_2]$ by the following formula:

$$[L_1 : L_2] = \frac{1}{2}\sum_{\nu=1}^{k}\Big(s(L_1(t_{\nu-1}), L_2(t_{\nu-1}), M_\nu) - s(L_1(t_\nu), L_2(t_\nu), M_\nu)\Big). \tag{6}$$

Example 2. $n = 1$. Let $\{e, f\}$ be a symplectic basis for E and consider the following family of symplectic transformations:

$$A(t) = \begin{pmatrix} \cos(\alpha t) & \sin(\alpha t) \\ -\sin(\alpha t) & \cos(\alpha t) \end{pmatrix},$$

$0 \le t \le T$. For $L := \mathrm{span}(e)$, $L^t := A(t)L = \mathrm{span}(e\cos(\alpha t) - f\sin(\alpha t))$ one finds

$$[L : L^t] = \begin{cases} k : & \alpha T = \pi k \\ \frac{1}{2} + k : & \pi k < \alpha T < \pi(k+1) \end{cases} \qquad (k \in \mathbb{Z}).$$

Example 3. Since the expressions $s(L_1, L_2, M)$ are locally constant in M as long as $M \cap L_i = \{0\}$, the above definition also makes sense if one considers a symplectic vector bundle over $[a,b]$ and replaces the L_i by Lagrangian subbundles. For instance, let N be a Lagrangian submanifold of T^*X and $\gamma : [a,b] \to N$ a continuous path on N. The *Maslov index* μ of γ is the intersection number of the tangent space $T_{\gamma(t)}N$ with the vertical polarization $V_{\gamma(t)}$:

$$\mu := [V_{\gamma(t)} : T_{\gamma(t)}N].$$

Proposition 4 (Properties of the intersection number).
(1) Antisymmetry: $[L_1 : L_2] + [L_2 : L_1] = 0$.
(2) Invariance: $[A(L_1) : A(L_2)] = [L_1 : L_2]$ *for all continuous paths* $A : [a,b] \to \mathrm{Sp(E)}$.
(3) $[L_1 : L_2] + \frac{1}{2}\dim(L_1(a) \cap L_2(a)) + \frac{1}{2}\dim(L_1(b) \cap L_2(b)) \in \mathbb{Z}$. *In particular,* $[L_1 : L_2]$ *is an integer if the intersections at the endpoints are transversal.*
(4) If $L_3 : [a,b] \to \Lambda(E)$ *is a third path,*

$$\begin{aligned}&[L_1 : L_2] + [L_2 : L_3] + [L_3 : L_1] \\ &\quad = \tfrac{1}{2}\Big(s(L_1(a), L_2(a), L_3(a)) - s(L_1(b), L_2(b), L_3(b))\Big). \qquad (7)\end{aligned}$$

(5) Consider the space of paths $L_1 \times L_2 : [a,b] \to \Lambda(E)^2$ *with given dimensions of the intersections at the endpoints.* $[L_1 : L_2]$ *labels the connected components of this space.*
(6) If $K(t)$ *is a continuous curve of isotropic subspaces contained in* L_1 *such that* $\dim(K \cap L_2)$ *is constant, the reductions* L_i^K *of* L_i *with respect to* K *are continuous, and*

$$[L_1 : L_2] = [L_1^K : L_2^K].$$

Proof. Properties 1–4 follow easily from the definition and proposition 1. Let us prove property 5. It is obvious that the intersection number is locally constant on the space in question. Conversely, suppose that $[L_1 : L_2] = [L_1' : L_2']$ for two paths in this space. We must show that they lie in the same connected component. By continuously deforming $L_1' \times L_2'$, we may assume that $L_1 = L_1'$ and that L_2 and L_2' coincide at the endpoints. Similarly, we can achieve that L_2 is constant. According to property 4, $[L_2 : L_2'] = 0$. This means that evaluating the Maslov class [11] of $\Lambda(E)$ on the closed path L_2' gives zero. But since the Maslov class generates $H^1(\Lambda(E))$, this proves that L_2' is homotopic to the constant path L_2. For property 6, observe that one may take the M_ν in the definition of the intersection number as continuous curves, and that it suffices to require $\dim(L_i(t) \cap M_\nu(t)) = \text{const.}$ on $[t_{\nu-1}, t_\nu]$. We hence take $M_\nu(t) = K(t) + K'_\nu(t)$ for suitable isotropic K'_ν and apply the reduction lemma. □

Using the intersection number, one arrives at a straightforward construction of the so-called *Leray index* [11,15]. Let $\pi : \tilde{\Lambda}(E) \to \Lambda(E)$ denote the universal covering of the Lagrange–Grassmann manifold. For $u_0, u_1 \in \tilde{\Lambda}(E)$, choose any path $u : [0,1] \to \tilde{\Lambda}(E)$ such that $u(0) = u_0$ and $u(1) = u_1$, and let $L(t) = \pi(u(t))$. Define the Leray index $m(u_0, u_1) \in \frac{1}{2}\mathbb{Z}$ by

$$m(u_0, u_1) = [L(t) : L(1)] = [L(t) : L(0)].$$

Proposition 4 guarantees that this is independent of the chosen path and immediately leads to the following statements:

Proposition 5 (Properties of Leray's index).

(1) For $L_i = \pi(u_i)$, Leray's formula holds:

$$m(u_1, u_2) + m(u_2, u_3) + m(u_3, u_1) = \tfrac{1}{2}s(L_1, L_2, L_3). \tag{8}$$

(2) For arbitrary lifts $u_i(\cdot)$ of Lagrangian curves $L_i(\cdot)$,

$$[L_1 : L_2] = m(u_1(a), u_2(a)) - m(u_1(b), u_2(b)). \tag{9}$$

(3) $m(u_1, u_2)$ is locally constant on the set of all u_1, u_2 with fixed $\dim(L_1 \cap L_2)$.

Conversely, properties 1 and 3 imply [9] that this definition of Leray's index is equivalent to the constructions in refs. [11,15].

Let $\tau : \widetilde{\mathrm{Sp}}(E) \to \mathrm{Sp}(E)$ denote the universal covering group of the symplectic group. Elements $\tilde{A}$ of the covering group can be identified with homotopy classes of paths $A(t)$ in $\mathrm{Sp}(E)$ connecting the identity to $A = \tau(\tilde{A})$. Recall that the graph

$$\Gamma_B := \{(Bx, x) | x \in E\}$$

of a symplectic transformation B in E is a Lagrangian subspace of $E \times E^-$, which is $E \oplus E$ with the symplectic form $\mathrm{pr}_1^* \omega - \mathrm{pr}_2^* \omega$. We hence obtain an index

$$\mu : \widetilde{\mathrm{Sp}}(E) \to \tfrac{1}{2}\mathbb{Z}, \quad \tilde{A} \mapsto [\Delta : \Gamma_{A(t)}], \tag{10}$$

where Δ is the graph of the identity, i.e. the diagonal in $E \times E^-$. (Equivalent indices are introduced in refs. [3] and [4].)

Proposition 6 (Properties of the index μ).

(1) $\mu(\tilde{A})$ is locally constant on the set of all $\tilde{A}$ with given $\dim(\ker(A - I))$.

(2) $\mu(\tilde{A}) + \frac{1}{2}\dim(\Gamma_A \cap \Delta) \in \mathbb{Z}$.

(3) $\mu(\tilde{A}^{-1}) = -\mu(\tilde{A})$.

(4) $\mu(S\tilde{A}S^{-1}) = \mu(\tilde{A})$ for all $S \in \mathrm{Sp}(E)$.

(5) Let $A(\cdot) : [0,1] \to \mathrm{Sp}(E)$ be any path representing $\tilde{A}$, and let $L, M \in \Lambda(E)$ be arbitrary. Then

$$\mu(\tilde{A}) = [M : A(t)L] + \tfrac{1}{2} s(\Delta, L \times M, \Gamma_A). \tag{11}$$

If $\ker(A - I)$ is symplectic and if L is A-invariant, the second term on the rhs vanishes.

(6) (See ref. [3].) Two elements of the set of all $\tilde{A}$ with $\ker(A - I) = \{0\}$ are in the same connected component if and only if they have the same index.

Proof. The first two statements are immediate from proposition 4 since $\ker(A - I) \cong (\Gamma_A \cap \Delta)$. Property 3 follows from $[\Delta : \Gamma_{A(t)^{-1}}] = [\Gamma_{A(t)} : \Delta]$. Property 4 is a special case of property 1 since the lhs is invariant if one connects S to the unit element. Equation (11) follows from proposition 4.4 and $[M : A(t)L] = [M \times L : \Gamma_{A(t)}]$. Assume now that L is invariant under A and that $E_1 := \ker(A - I)$ is symplectic. Since we may decompose $E = E_1 \oplus E_1^{\omega}$, it is sufficient to study the cases $E_1 = E$ and $E_1 = \{0\}$. The first case is equivalent to $\Gamma_A = \Delta$ and hence trivial. In the second case, one has $\Gamma_A \cap \Delta = \{0\}$. Observe that, due to proposition 4.4, $[M : A(t)L]$ is independent of the choice of M. It is therefore sufficient to prove $s(\Delta, L \times M, \Gamma_A) = 0$ for $M = L$.

Since $K := (L \times L) \cap \Delta$ and Γ_A have trivial intersection, $\Gamma_A^K \cong \Gamma_A \cap K^{\omega}$ and $\Gamma_A \cap (L \times L)$ both have dimension $n = \frac{1}{2}\dim(E)$ and must therefore be identical. This proves $\Gamma_A^K = (L \times L)^K$ and thus

$$s(\Gamma_A, \Delta, L \times L) = s(\Gamma_A^K, \Delta^K, (L \times L)^K) = 0.$$

Finally, property 6 is equivalent to theorem 1 and lemma 1.7 in ref. [3] after one has identified $\mu(\tilde{A})$ with the index for exponential paths constructed there, which is done by a glance at the following examples. □

Example 7. We can use the above theorem to compute $\mu(\tilde{A})$ in example 2. If $\alpha T = \pi k$, $A(T) = \pm I$, so the result from example 2 gives for arbitrarily chosen L

$$\mu(\tilde{A}) = [L : A(t)L] = k.$$

If $2\pi k < \alpha T < 2\pi(k+1)$, it follows that $\mu(\tilde{A}) = 2k + 1$ because $\mu(\tilde{A})$ is constant on this set.

Example 8. Assume $A(t) = \exp(tS)$ for some $S \in \mathrm{sp}(E)$ which has no purely imaginary eigenvalues. The stable subspace $L \subset E$ for $A(t)$ is Lagrangian, hence $\mu(\tilde{A}) = [L : A(t)L] = [L : L] = 0$.

Example 9. In a canonical basis for E, let

$$A(t) = \begin{pmatrix} 1 & tP \\ 0 & 1 \end{pmatrix}$$

for some symmetric P. Let $L = \operatorname{span}\{e_i\}$ and $M = \operatorname{span}\{f_i\}$. Since L is invariant, the proposition gives

$$\begin{aligned}\mu(\tilde{A}) &= \tfrac{1}{2}s(\Delta, L \times M, \Gamma_A)\\ &= \tfrac{1}{2}\big(s(M \times M, L \times M, \Gamma_A) + s(\Delta, M \times M, \Gamma_A)\\ &\quad - s(\Delta, M \times M, L \times M)\big)\\ &= \tfrac{1}{2}s(M, L, A(M)) = \tfrac{1}{2}\operatorname{sgn}(T)\operatorname{sgn}(P),\end{aligned}$$

where we have applied the reduction lemma with $K = \Delta \cap (L \times L)$ and $K = \{0\} \times M$.

Proposition 10. *The equation* $w(A_1, A_2) = s(\Delta, \Gamma_{A_1}, \Gamma_{A_1A_2})$ *defines a cocycle on* $\operatorname{Sp}(E)$, *i.e.*,

$$w(A_1A_2, A_3) + w(A_1, A_2) = w(A_1, A_2A_3) + w(A_2, A_3).$$

Considered as a cocycle on $\widetilde{\operatorname{Sp}}(E)$, *it cobounds* 2μ:

$$\mu(\tilde{A}_1\tilde{A}_2) - \mu(\tilde{A}_1) - \mu(\tilde{A}_2) = \tfrac{1}{2}w(A_1, A_2). \tag{12}$$

Proof. Of course, it suffices to prove the second assertion. Choosing a fixed preimage $\tilde{\Delta}$ of Δ, we have an identification of $\widetilde{\operatorname{Sp}}(E)$ as a subset of $\tilde{\Lambda}(E \times E^-)$. Equation (12) then follows from Leray's formula and $m(\tilde{A}_1\tilde{A}_2, \tilde{A}_1) = m(\tilde{A}_2, \tilde{\Delta}) = \mu(\tilde{A}_2)$. □

2. The Conley–Zehnder index of periodic trajectories

Let $\gamma : \mathbb{R} \to Q$ be a periodic trajectory of period T for the classical mechanical system (Q, ω, H). Denote its orbit $\gamma(\mathbb{R})$ by $\gamma^\sharp$ and its fundamental period by $T^\sharp$, i.e., $T = kT^\sharp$ for some nonzero integer k.

The flow $F^t = \exp(tX_H)$ generates a family of canonical transformations in the symplectic vector bundle $TQ|_{\gamma^\sharp}$. Let $\mathcal{E}^1$ be the reduced bundle with respect to $K = \operatorname{span}(X_H)|_{\gamma^\sharp}$. Let $q = \gamma(0)$ be some reference point and

$$P(t) : \mathcal{E}^1_q \to \mathcal{E}^1_{F^t(q)} \tag{13}$$

be the induced flow. $P(T) : E^1_q \to E^1_q$ is called (linear) Poincaré map, and the periodic trajectory is called nondegenerate if $P(T) - I$ is invertible. As already mentioned, nondegenerate periodic orbits are contained in two-dimensional symplectic "orbit cylinders", hence $\mathcal{E}^1$ can be regarded as the symplectic orthogonal to the tangent bundle $\mathcal{E}^2$ of the orbit cylinder.

Assume that γ is nondegenerate and that we are given some distinguished homotopy class of Lagrangian subbundles L of $TQ|_{\gamma^\sharp}$. [This is equivalent to specifying a lift of the bundle $\Lambda(TQ|_{\gamma^\sharp})$ of Lagrangian Grassmannians to a bundle $\tilde{\Lambda}(TQ|_{\gamma^\sharp})$ of their universal coverings. Indeed, the homotopy class of the closed curve $L : \gamma^\sharp \to \Lambda(TQ)|_{\gamma^\sharp}$ is then uniquely determined by the requirement that its lifts to $\tilde{\Lambda}(TQ|_{\gamma^\sharp})$ should be closed.] Consider for instance one of the following cases:

(1) Q is a cotangent bundle and L is the vertical polarization. [More generally, one can consider situations where one has a distinguished lift $\tilde{\Lambda}(TQ) \to \Lambda(TQ)$. Since the structure group of $\Lambda(TQ)$ is $G = \mathrm{Sp}(\mathbb{R}^{2n})/\{I, -I\}$ and $\pi_1(G) = \mathbb{Z}$, there is an obstruction $\beta \in H^2(Q, \mathbb{Z})$ to the existence of such a lift, and uniqueness is equivalent to $H^1(Q, \mathbb{Z}) = \{0\}$.]

(2) The orbit $\gamma^\sharp$ is contractible.

As announced in the introduction, we want the index $\mathrm{ind}(\gamma, L)$ to be stable under small perturbations of the Hamiltonian system. More precisely, we require

(A1) The index depends only on L and on $P(\cdot)$, and is invariant under homotopies of $P(\cdot)$ which leave $(P(T) - I)$ invertible.

(A2) If $TF^T(M) = M$ for some $M \in \Lambda(T_qQ)$, $\mathrm{ind}(\gamma, L) = [L(\gamma(t)) : TF^tM]$.

Up to homotopy, there is unique trivialization $\mathcal{E}^2 \to \gamma^\sharp \times E^2$ mapping K to a constant bundle. Let us also choose any trivialization of $\mathcal{E}^1$, so that $P(t)$ becomes a curve in $\mathrm{Sp}(E^1)$.

Let M be any fixed Lagrangian subspace of T_qQ, regarded as a Lagrangian subbundle of $TQ_{\gamma^\sharp}$ via the trivialization. It is obvious that

$$\mathrm{ind}(\gamma, L) := [L(\gamma(t)) : M] + \mu(\widetilde{P(T)}) \tag{14}$$

is independent of the choice of q, M and the trivialization of $\mathcal{E}^2$. We check that it is also independent of the trivialization of $\mathcal{E}^1$. Each change of trivialization corresponds to a continuous map $S : \gamma^\sharp \to \mathrm{Sp}(E^1)$. Writing $S_t = S(\gamma(t))$, this replaces $P(t)$ by $P_S(t) = S_tP(t)S_0^{-1}$ and maps M to $M_S(t) = (S_t \times I_{E_2})M$. We find

$$\begin{aligned}[L(\gamma(t) : M_S(t)] - [L(\gamma(t)) : M] &= [M_S(t) : M] = [M_S(t) : M_S(0)] \\ &= \mu(\tilde{S}_T) - \mu(\tilde{S}_0),\end{aligned}$$

which according to proposition 6 is the same as $\mu(\widetilde{P_S(T)}) - \mu(\widetilde{P(T)})$.

Proposition 11. *The index* $\mathrm{ind}(\gamma, L)$ *satisfies (A1), (A2) and is uniquely determined by this property.*

Proof. It is clear from the definition that (A1) is fulfilled. To check (A2), assume that $M \subset T_qQ$ is TF^T-invariant. Since $[L(\gamma(t) : TF^tM]$ does not depend on the choice of the invariant M, we may replace M by $M^1 \oplus K$, where $M^1 \subset E^1$ is the reduction of M with respect to K. Using the above trivialization, property (A2) thus follows from

$$[L(\gamma(t)) : TF^t M] - [L(\gamma(t) : M] = [M : TF^t M]$$
$$= [M^1 : P(t)M^1] = \mu(\widetilde{P(T)}).$$

On the other hand, lemma 1.7. of ref. [3] shows that one can deform $P(\cdot)$ such that all eigenvalues of $P(T)$ are real. In that case, it is possible to construct an invariant Lagrangian subspace M of T_qQ: Pick an eigenvector for TF^T, observe that its ω-orthogonal complement is invariant as well, pick an eigenvector for the induced map on the reduced space and so on. The corresponding Lagrangian subbundle $M(\gamma(t)) = TF^t(M)$ of TQ_{γ^t} is thus invariant, which shows that the index is uniquely determined by (A2). □

We now give another explicit expression for the index, which does not require any trivialization.

Proposition 12. *For arbitrary Lagrangian subspaces $M \in \Lambda(T_qQ)$, the following formula is valid:*

$$\begin{aligned} \operatorname{ind}(\gamma, L) = {} & [L(\gamma(t)) : TF^t(M)] \\ & + \tfrac{1}{2} s(\Delta, L_q \times M, \Gamma_{TF^T}) + \tfrac{1}{2}\operatorname{sgn}(\partial T/\partial E). \end{aligned} \tag{15}$$

Proof. Abbreviate $\Gamma_{TF^t} = \Gamma_t$ and write $\Gamma_t = \Gamma_t^1 \times \Gamma_t^2$, $\Delta = \Delta^1 \times \Delta^2$. From proposition 6.3,

$$\begin{aligned} \mu(\widetilde{P(T)}) &= [\Delta^1 : \Gamma_t^1] = [\Delta : \Gamma_t] - [\Delta^2 : \Gamma_t^2] \\ &= [M : TF^t M] + \tfrac{1}{2} s(\Delta, M \times M, \Gamma_T) - \tfrac{1}{2} s(\Delta^2, K \times K, \Gamma_T^2). \end{aligned} \tag{16}$$

According to example 9, the third term is $\frac{1}{2}\operatorname{sgn}(\partial T/\partial E)$. The other terms can be combined with $[L(\gamma(t) : M]$ using proposition 4.3 and

$$s(\Delta, M \times M, \Gamma_T) + s(L_q, M, TF^T(M)) = s(\Delta, L_q \times M, \Gamma_T)$$

to yield the final result (15). □

If the periodic orbit is of hyperbolic type, i.e., if the Poincaré map has no eigenvalues on the unit circle, the stable and unstable manifolds are Lagrangian. In particular, (A2) shows that in this case the index behaves additively under multiple traversals: $\operatorname{ind}(k\gamma, L) = k \operatorname{ind}(\gamma, L)$. However, this conclusion is wrong in general. Even if the multiply traversed orbit is still nondegenerate, the behaviour of the index is determined by the difference

$$\mu(\widetilde{P(kT)}) - k\mu(\widetilde{P(T)}) = \frac{1}{2}\sum_{r=2}^{k-1} s(\Delta, \Gamma_{P(T)}, \Gamma_{P(rT)}),$$

which is usually nonzero, see for instance example 7. The large k behaviour of such expressions in terms of the conjugacy class of $P(T)$ was studied in great detail by Cushman and Duistermaat [4].

3. Proof of $\sigma_\gamma = \mathrm{ind}(\gamma, V^X)$

The aim of this section is to prove:

Theorem 13. *The Maslov phase σ_γ appearing in the trace formulas (1), (2) is equal to the index* $\mathrm{ind}(\gamma, V^X)$, *where V^X is the vertical polarization of T^*X. In particular, σ_γ is the winding number of the stable manifold if the orbit is hyperbolic.*

Recall from ref. [8] that eq. (1) originates from an application of the principal symbol calculus for Fourier integrals. A similar framework, with Fourier integrals replaced by oscillatory integrals [6,16,19], can also be used to derive eq. (2). (The major difference is that the corresponding Lagrangian manifolds are usually not conical, but this does not lead to special complications.) The integrals to be composed are the Schwartz kernel $\delta(x,y)$ of the identity and the Schwartz kernel $U(t,x,y)$ of the unitary group ("$\mathrm{tr}\, U(t) = \int\int U(t,x,y)\delta(x,y)$"). As oscillatory and Fourier integrals, respectively, the former is associated to the conormal bundle of the diagonal

$$\Gamma_\delta = \{(x,\xi;x,-\xi)\} \subset T^*(X \times X),$$

the latter to the canonical relation belonging to the flow F^t:

$$\Gamma_U = \{(t,\tau;x,\xi;y,-\eta) | F^t(y,\eta) = (x,\xi),\ \tau = -H(x,\xi)\}. \qquad (17)$$

Note that Γ_U is swept out from $\{(0,\tau)\} \times \Gamma_\delta$ by the flow of the extended Hamiltonian

$$\mathcal{H}(t,\tau;x,\xi;y,\eta) = \tau + H(x,\xi).$$

If the composition is clean in the sense of ref. [8], section 5, $\mathrm{tr}(U)$ is an oscillatory integral associated to

$$\begin{aligned}\mathcal{P} :&= \Gamma_U \circ \Gamma_\delta \\ &= \{t,\tau \mid \exists (x,\xi) \in T^*X : F^t(x,\xi) = (x,\xi), H(x,\xi) = -\tau\}. \qquad (18)\end{aligned}$$

The spectral density $g(E)$ is finally obtained by taking the Fourier transform of $\mathrm{tr}(U(t))$. It is hence an oscillatory integral associated to the image of $\mathcal{P}$ under the canonical transformation $(t,-E) \mapsto (E,t)$. From the known principal symbols of δ and U, the composition rule now yields the principal symbol of $\mathrm{tr}(U)$ and $g(E)$ and thus eqs. (1) and (2). For us, it suffices to describe the Maslov part of this composition rule and how it gives rise to σ_γ.

We start by recalling Hörmander's construction of *Maslov's principal bundle* $\mathcal{M}$ over the Lagrangian Grassmannian $\Lambda(E)$ of a symplectic vector space E. Let $L_1 \in \Lambda(E)$ be fixed. According to proposition 1, the expressions

$$\tfrac{1}{2}\big(s(L_1,L_2,M_1) - s(L_1,L_2,M_2)\big) = \tfrac{1}{2}\big(s(M_1,M_2,L_2) - s(M_1,M_2,L_1)\big) \quad (19)$$

are locally constant and integer-valued as long as $L_i \cap M_j = \{0\}$, $i, j = 1, 2$. Using them as transition functions, a section of $\mathcal{M}$ over an open subset $\mathcal{U} \subset \Lambda(E)$ can be regarded as a function

$$\phi : \mathcal{U} \times \Lambda(E) \to \mathbb{Z} \tag{20}$$

such that $\phi(L_2, M) - \frac{1}{2}s(L_1, L_2, M)$ is independent of M and $\phi(L_2, M)$ is continuous on the set defined by $M \cap L_i = \{0\}$.

In order to avoid undue complications, we will enlarge the structure group to $\frac{1}{2}\mathbb{Z}$ (call that bundle $\mathcal{M}'$), which amounts to replacing $\mathbb{Z}$ by $\frac{1}{2}\mathbb{Z}$ in (20). One special feature of $\mathcal{M}'$ is that each point $L_2^0 \in \Lambda(E)$ determines the germ of a trivialization. Indeed, let $\mathcal{U}$ be some contractible neighbourhood of L_2^0, take $\phi(L_2^0, M) = \frac{1}{2}s(L_1, L_2^0, M)$ and use parallel transport on $\mathcal{M}'$. This yields

$$\phi(L_2, M) = \tfrac{1}{2}s(L_1, L_2, M) + [L_1 : L_2(t)], \tag{21}$$

where $L_2(t)$ is any path in $\mathcal{U}$ leading from L_2^0 to L_2.

Maslov's principal bundle over a Lagrangian submanifold N of T^*X is defined in a similar way, letting L_1 be the vertical polarization and L_2 the tangent bundle of N. Suppose now that N_2, N_1 are Lagrangian submanifolds of $T^*(X_3 \times X_2)$ and $T^*(X_2 \times X_1)$, respectively, equipped with sections ϕ_i of their Maslov bundles $\mathcal{M}_i'$. Let $S \subset T^*(X_2 \times X_2)$ be the conormal bundle of the diagonal. Then $N_2 \circ N_1$ is, by definition, the image of $(T^*X_3 \times S \times T^*X_1) \cap (N_2 \times N_1)$ under the symplectic reduction $\rho : T^*X_3 \times S \times T^*X_1 \to T^*(X_3 \times X_1)$, and it is an immersed Lagrangian manifold if the intersection is clean (cf. ref. [1]). The composed section $\phi_2 \circ \phi_1$ is defined by

$$\phi_2 \circ \phi_1(W_p) = (\phi_2 \times \phi_1)((T_z\rho)^{-1}(W_p))$$

for $W_p \in \Lambda(T_p(T^*(X_3 \times X_1)))$ and arbitrary $z \in \rho^{-1}(p)$.

We now return to the particular case under consideration. In the sequel, $V^R = \mathrm{span}(\partial/\partial E)$ and V^X denote the vertical polarizations in $T^*\mathbb{R}$ and T^*X, respectively. By abuse of notation, all the other vertical polarizations that appear will simply be denoted by V. The canonical trivialization of the Maslov bundle over Γ_δ is defined by

$$\phi_\delta(W_z) = \tfrac{1}{2}s(V_z, T_z\Gamma_\delta, W_z) .$$

Parallel transport along the solution curves κ of $X_{\mathcal{H}}$ induces a trivialization of the Maslov bundle over Γ_U:

$$\phi_U(W_{\kappa(T)}) = [V_{\kappa(t)} : T_{\kappa(t)}\Gamma_U] + \tfrac{1}{2}s(V_{\kappa(T)}, T_{\kappa(T)}\Gamma_U, W_{\kappa(T)}).$$

Using proposition 4.6, one easily finds that the first term is equal to $[V^X : TF^t(V^X)]$. At noncaustic points, i.e., where Γ_U is transversal to the vertical polarization, the second term is just the canonical trivialization described above. We have hence recovered the Maslov phase in the semiclassical van Vleck formula for the unitary group as the transition function to the canonical trivialization.

Assume now that $(T,-E) \in \mathcal{P}$ corresponds to a nondegenerate periodic trajectory, $T \neq 0$. Using the above rule, we can compute the induced section $\phi_U \circ \phi_\delta$ of the Maslov bundle over $\mathcal{P}$. The final step of taking the Fourier transformation is another composition with an oscillatory integral, associated to the graph of the canonical transformation $(t,-E) \mapsto (E,t)$. On the symbol level, this just exchanges the vertical and the horizontal polarizations of $T^*\mathbb{R}$. Finally, the phase σ_γ appears as the transition function from the composed sections of the Maslov bundle to the canonical trivialization (21) at (E,T). Evaluating this on $Z := \operatorname{span}(\partial/\partial t)$, we obtain

$$\begin{aligned}\sigma_\gamma &= (\phi_U \circ \phi_\delta)(Z) = (\phi_U \times \phi_\delta)(Z \times TS) \\ &= [V^X : TF^T(V^X)] + \tfrac{1}{2}s(V^R \times V, T\Gamma_U \times T\Delta, Z \times TS).\end{aligned}$$

(For convenience of notation, the base points will be omitted from now on.) From the cocycle identity and the reduction lemma, one finds

$$\begin{aligned}&s(V^R \times V, T\Gamma_U \times T\Delta, Z \times TS) - s(V^R \times V, T\Gamma_U \times T\Delta, V^R \times TS) \\ &\quad = s(V^R, Z, T\mathcal{P}) = \operatorname{sgn}(\partial T/\partial E).\end{aligned}$$

Similarly,

$$\begin{aligned}s(V^R \times V, T\Gamma_U \times T\Delta, V^R \times TS) &= s(V, \Gamma(TF^T) \times T\Delta, TS) \\ &= s(V \times T\Delta, \Gamma(TF^T) \times T\Delta, TS) - s(V, V \times T\Delta, TS) \\ &\quad + s(V, V \times T\Delta, \Gamma(TF^T) \times T\Delta) \\ &= s(V, \Gamma(TF^T), T\Delta) - s(V, T\Delta, V) + s(V, V, \Gamma(TF^T)) \\ &= s(V, \Gamma(TF^T), T\Delta),\end{aligned}$$

hence

$$\sigma_\gamma = [V^X : TF^t(V^X)] + \tfrac{1}{2}s(V, \Gamma(TF^T), T\Delta) + \tfrac{1}{2}\operatorname{sgn}(\partial T/\partial E). \tag{22}$$

Comparing this to proposition 12, the theorem follows.

Appendix A. Maslov's bundle in the complex case

It is interesting to note that the expressions (19) may also be looked upon as "complex phase changes". To explain this point of view, which is motivated by ref. [17], let L_i, M_j be complex Lagrangian subspaces of $E_\mathbb{C} = E \otimes \mathbb{C}$, $L_i \cap M_j = \{0\}$. The projection of L_1 onto L_2 along M_j induces an isomorphism $\pi_j : \Lambda^n L_1 \to \Lambda^n L_2$. Define

$$\tau(L_1, L_2, M_1, M_2) = \pi_1 \circ \pi_2^{-1} \in \mathbb{C}^*. \tag{A.1}$$

τ satisfies an obvious cocycle condition in the M's and may therefore be used to define, for L_1 fixed, a principal $\mathbb{C}^*$-bundle over $\Lambda(E_\mathbb{C})$. We now restrict attention to the case L_i positive semidefinite, M_j negative semidefinite. (A Lagrangian subspace L of $E_{\mathbb{C}}$ is called positive/negative semidefinite, if $G(x,y) = \frac{1}{i}\omega(\bar{x},y)$ is positive/negative semidefinite on L.) Since the set of negative semidefinite Lagrange subspaces is simply connected, there is a unique continuous choice of $\arg\tau$ such that

$$\arg\tau(L_1, L_2, M, M) = 0.$$

This gives rise to a principal $\mathbb{R}$-bundle over the set of positive semidefinite Lagrangian subspaces. We claim that Maslov's bundle may, in a sense, be regarded as the restriction of this bundle to $\Lambda(E) \hookrightarrow \Lambda(E_C)$:

Proposition 14. *If L_i, M_j are complexifications of real Lagrange subspaces (which we continue to denote by L_i, M_j),*

$$\tfrac{1}{2}\big(s(L_1,L_2,M_1) - s(L_1,L_2,M_2)\big) = -\frac{1}{\pi}\arg\tau(L_1,L_2,M_1,M_2). \qquad \text{(A.2)}$$

Proof. Both sides are invariant under continuous changes of L_i which leave them real and transversal to M_j. We may therefore suppose that $L_1 \cap L_2 = \{0\}$ and choose a symplectic basis $(e_1,\ldots,e_n,f_1,\ldots,f_n)$ such that L_1 is spanned by the e's and L_2 by the f's. Similarly, we may suppose that the M_j are spanned by vectors $g_k = e_k \pm f_k$, and it suffices to investigate the flip of one sign. For this, it is sufficient to consider the case $n = 1$. Let $g(t) = e + \mathrm{e}^{-\mathrm{i}\pi t} f$ and $M(t) = \mathrm{span}(g(t))$. The projection of e onto L_2 along $M(t)$ is $\mathrm{e}^{-\mathrm{i}\pi t} f$, thus

$$\arg\tau(L_1,L_2,M(0),M(t)) = \pi t.$$

On the other hand,

$$s(L_1,L_2,M(0)) - s(L_1,L_2,M(1)) = -1-(-1) = -2. \qquad \square$$

We now choose a real symplectic basis for E such that L_α, M_β are transversal to $\mathrm{span}(e_1,\ldots,e_n)$. In this basis, they are spanned by vectors $f_i + \sum_j A^\alpha_{ij} e_j$ and $f_i + \sum_j B^\beta_{ij}$, respectively, where A^α, B^β are symmetric. One finds

$$\tau(L_1,L_2,M_1,M_2) = \frac{\det(A^1 - B^1)\,\det(A^2-B^2)}{\det(A^2-B^1)\,\det(A^1-B^2)}. \qquad \text{(A.3)}$$

This leads to several properties of τ:

$$\tau(L_1,L_2,M_1,M_2) = \tau(M_1,M_2,L_1,L_2), \qquad \text{(A.4)}$$

$$\tau(\overline{L}_1,\overline{L}_2,\overline{M_1},\overline{M_2}) = \overline{\tau(L_1,L_2,M_1,M_2)}, \qquad \text{(A.5)}$$

$$\tau(L,M_1,M_2,M_3)\,\tau(L,M_2,M_3,M_1)\,\tau(L,M_3,M_1,M_2) = (-1)^{3n}, \qquad \text{(A.6)}$$

whenever the transversality conditions are fulfilled. If L_α and M_β are positive and negative semidefinite, respectively, we obtain from eqs. (A.4), (A.5):

$$\arg(L_1,L_2,M_1,M_2) = -\arg(\overline{M}_1,\overline{M}_2,\overline{L}_1,\overline{L}_2). \qquad \text{(A.7)}$$

Together with proposition 14 this gives a proof of the cocycle identity 1.1 for the case $L_\alpha \cap M_\beta = \{0\}$.

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References

[1] R. Abraham and J. Marsden, *Foundations of Mechanics* (Addison-Wesley, 1978).

[2] R. Brummelhuis and A. Uribe, A semi-classical trace formula for Schrödinger operators, Commun. Math. Phys. 136 (1991) 567–584.

[3] C. Conley and E. Zehnder, Morse-type index theory for flows and periodic solutions for Hamiltonian equations, Commun. Pure Appl. Math. 37 (1984) 207–253.

[4] R. Cushman and J.J. Duistermaat, The behaviour of the index of a periodic linear Hamiltonian system under iteration, Adv. Math. 23 (1977) 1–21.

[5] P. Dazord, Invariantes homotopiques attachés aux fibrés symplectiques, Ann. Inst. Fourier Grenoble 29(2) (1979) 25–78.

[6] J.J. Duistermaat, Oscillatory integrals, Lagrange immersions and the unfolding of singularities, Commun. Pure Appl. Math. 27 (1974) 207–281.

[7] J.J. Duistermaat, On the Morse index in variational calculus, Adv. Math. 21 (1976) 173–195.

[8] J.J. Duistermaat and V. Guillemin, The spectrum of positive elliptic operators and periodic bicharacteristics, Inv. Math. 29 (1975) 39–79.

[9] M.A. de Gosson, La définition de l'indice de Maslov sans hypothèse de transversalité, C. R. Acad. Sci. Paris 310 (1990) 279–282.

[10] M.C. Gutzwiller, Periodic orbits and classical quantization conditions, J. Math. Phys. 12(3) (1971) 343–358.

[11] V. Guillemin and S. Sternberg, *Geometric Asymptotics*, Mathematical Surveys, Vol. 14 (Am. Math. Soc., 1977).

[12] V. Guillemin and A. Uribe, Reduction and the trace formula, J. Diff. Geom. 32 (1990) 315–347.

[13] L. Hörmander, Fourier integral operators I, Acta Math. 127 (1971) 79–183.

[14] P. Libermann and C.M. Marle, *Symplectic Geometry and Analytical Mechanics* (Reidel, Dordrecht, 1987).

[15] G. Lion and M. Vergne, *The Weil Representation, Maslov Index and Theta Series*, Progress in Mathematics, Vol. 6 (Birkhäuser, Basel, 1980).

[16] E. Meinrenken, Semiclassical principal symbols and Gutzwiller's trace formula, preprint, Freiburg (1991), to appear in Rep. Math. Phys.

[17] A. Melin and J. Sjöstrand, Fourier integral operators with complex valued phase functions, Lecture Notes in Math. 459 (1974) 120–223.

[18] J.M. Robbins, Maslov indices in the Gutzwiller trace formula, Nonlinearity 4 (1991) 343–363.

[19] D. Robert, *Autour de l'approximation semi-classique* (Birkhäuser, Basel, 1987).